Math 259A Lecture 22 Notes

Daniel Raban

November 20, 2019

1 The Hyperfinite II_1 Factor

1.1 Construction

Here is another example of a II_1 factor.

Consider the algebra $R^0 = \bigotimes_{n=1}^{\infty}(M_2(\mathbb{C}), \mathrm{tr})_n$ (this is just algebraic). This is

$$M_2 \xrightarrow{x \mapsto x \otimes 1} M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \xrightarrow{x \mapsto x \otimes 1} M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \longrightarrow \cdots$$

and we take the limit. These inclusions look like

$$x \mapsto \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}.$$

The elements of an infinite tensor product are elements that look like

$$x_1 \otimes x_2 \otimes x_3 \otimes \cdots \otimes x_n \otimes 1 \otimes 1 \otimes \cdots$$

for some n.

The algebra R^0 has a trace state given by

$$\tau(x_1 \otimes \cdots \otimes x_n \otimes 1 \otimes \cdots) = \mathrm{tr}(x_1)\,\mathrm{tr}(x_2) \cdots \mathrm{tr}(x_n).$$

This is consistent if we take tr on $M_2(\mathbb{C})$ to be normalized. So $\mathrm{tr}_{M_2(\mathbb{C})}(x) = \mathrm{tr}_{M_{2^{n+1}}(\mathbb{C})}(x)$ via the above inclusions.

R^0 is a *-algebra with the operator norm $\|x\| = \|x\|_{M_{2^n}(\mathbb{C})}$ if $x = x_1 \otimes \cdots \otimes x_n \otimes 1 \otimes \cdots$. This is consistent with the inclusions because the operator norm satisfies $\|x \oplus y\| = \max\{\|x\|, \|y\|\}$. This norm satisfies the C^* axiom: $\|x^*x\| = \|x\|^2$. Thus, $(R_0, \|\cdot\|) := \overline{(R^0, \|\cdot\|)}^{\|\cdot\|}$ is a C^*-algebra, and τ extends to a trace state on R_0 (exercise).

Proposition 1.1. *Let $x \in R^0$. If $x \in M_{2^n}(\mathbb{C})$, then*

$$\tau(x) = \int_{U(M_{2^n}(\mathbb{C}))} uxu^* \, du,$$

where the integral is with respect to Haar measure on the unitary group $U(M_{2^n}(\mathbb{C}))$.

Remark 1.1. Since this is in finite dimensions, this is Riemann integral, uniformly convergent in the operator norm

Proof. Call this integral $\Phi(x) \in M_{2^n}(\mathbb{C})$. By the invariance of Haar measure,

$$\Phi(x) = \Phi(u_0 x u_0^*) = u_0 \Phi(x) u_0^*.$$

This implies that $\Phi(x)u_0 = u_0\Phi(x)$ for all unitary u_0. So $\Phi(x) \in M_{2^n}' \cap M_{2^n} = \mathbb{C}$.

This Φ has the properties of the trace, so by uniqueness of the trace, $\Phi(x) = \tau(x)1$. $\square$

Proposition 1.2. *Let $x \in R^0$. Then $\tau(x^*x) = 0$ if and only if $x = 0$.*

Apply the GNS construction for (R_0, τ) to get the representation $(\pi_\tau, H_\tau, \xi_\tau = \hat{1})$; recall that $H_\tau = \overline{R_0}^{\|\cdot\|_\tau}$. So $x \mapsto \pi_\tau(x)$, which is left-multiplication by x on $\widehat{R_0} = H_\tau^0$, which contains $\widehat{R^0}$ as a dense subset. Also, we have $\tau(x) = \langle x\hat{1}, \hat{1}\rangle_{H_\tau}$.

π_τ is isometric because it is isometric on each $M_{2^n}(\mathbb{C})$ (since it is an injective morphism of C^*-algebras).

Definition 1.1. The **hyperfinite II_1 factor** (R, τ) is $\overline{\pi_\tau(R_0)}^{\text{wo}}$, endowed with the trace state $\tau(x) = \langle x\hat{1}, \hat{1}\rangle$.

This is a von Neumann algebra.

1.2 R is a II_1 factor

Proposition 1.3. *τ is faithful on R ($\tau(x^*x) = 0 \iff x = 0$) if and only if ξ_τ is separating for R.*

Proof. ($\implies$): $\tau(x^*x) = \|x\xi_\tau\|_{H_\tau}^2$. We have $R = \overline{\pi_\tau(R^0)}^{\text{wo}}$m but we could have taken right multiplication in the GNS construction. Also λ and ρ, left and right multiplication, commute. So $\rho(y)(x\hat{1} =)$ for all R^0. Thus, $\rho(y)(x\hat{1}) = 0$ for all $y \in R^0$, so $[R, \rho(R^0)] = 0$. So if $x\hat{1} = 0$, then $\rho(y)x(\hat{1}) = 0 = x(\rho(y)\hat{1}) = xy = x(\hat{y}) = 0$. This implies that $x(H_\tau) = 0$, so $x = 0$. $\square$

τ on R is a faithful trace. In particualar, R is finite.

Proposition 1.4. *(R, τ) is a II_1 factor.*

Proof. Assume $z \in (Z(M))_1$. Then there exists some $x_i \in (R_0)_1$ such that $\pi_\tau(x_i) \xrightarrow{\text{so}} z$ by Kaplansky's theorem. We have

$$\|\pi_\tau(x_i)\hat{1} - z(\hat{1})\|_{H_\tau} \to 0 \iff \|x_i - z\|_\tau \to 0,$$

where $\|x_i - z\|_\tau = \|ux_ix^* - z\|_\tau$ for any unitary u. But for each fixed i, if $x \in M_{2^{n_i}}(\mathbb{C})$, then $\|\int ux_iu^* - z\,du\|_\tau \leq \|ux_iu^* - z\|_\tau$ for all unitary u. But the left hand side is $\|\tau(x_i) - z\|_\tau$. Therefore, $\|z - c\hat{1}\|_\tau = 0$. But τ is faithful, so $z = c\hat{1}$ is a scalar. $\square$